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LETTER TO THE EDITOR

A remark on possible violations of the Pauli principle†

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Abstract. Recently Ignatiev and Kuzmin developed a theoretical model to implement small violations of the Pauli principle. We remark that the Pauli principle is unique among discrete symmetries, and that in consequence any apparent violation actually signals new physical degrees of freedom, with no violation of the Pauli principle. We construct an algebraic model which incorporates the algebra of Ignatiev and Kuzmin and leads to the same apparent violations, yet preserves the Pauli principle. Our algebra is that of a Jordan pair, an algebraic concept which provides a natural framework for structures that do not assume bilinear commutation relations.

The construction of a theoretical framework [1] in which arbitrarily small violations of the Pauli principle may exist has recently been discussed [2, 3]. Let us assume that, for purposes of discussion, such a violation has been found; what would this mean?

To answer this question it is helpful to consider the historical background in the development of the Pauli principle. Bohr in his *Aufbauprinzip* for atomic structure assumed *implicitly* the equivalent of the Pauli principle. Pauli, as is very well known, made the concept explicit by requiring distinct quantum numbers for each electron.

It was Dirac who recognised that the fundamental underlying concept was that of a discrete involutary symmetry: the exchange of equivalent particles. Discrete symmetries, however—unlike continuous symmetries (such as rotational symmetry)—need not be *quantal* symmetries, as Dirac [4] pointed out in 1949. Since the well known violations of the discrete symmetries of parity and time-reversal invariance verify Dirac's remark, there would seem to be no objection in principle to a similar violation of the Pauli principle. We will argue, however, that the symmetry underlying the Pauli principle cannot be violated by an arbitrarily small amount; any such apparent violation only signals new degrees of freedom¶.

At the classical level, the concept of nearly identical particles leads to the Gibbs paradox: a gas containing equally two species of particles which differ infinitesimally from being identical has a macroscopically large entropy change as the infinitesimal difference vanishes. Landé in particular emphasised the necessity of quantum mechanics to eliminate such unphysical discontinuities [6, 7].

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¶ In a similar way it has been remarked [5] that parastatistics is equivalent to ordinary statistics with additional internal degrees of freedom.

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Quantum mechanically, as Landé pointed out, the particles would have state vectors consisting of at least two distinct states mixed infinitesimally. The resulting thermodynamic ensemble shows no discontinuity as the infinitesimal mixing parameter goes to zero.

Let us analyse this resolution of the paradox further. The key point is to recognise that the concept of identity has been transferred from 'classical particle' to quantal state. In other words, the concept of classical particles which differ continuously from each other can only avoid unacceptable logical paradoxes by having the particles possess at least two non-identical quantal states lying in the same Hilbert space. Expressed still differently, but equivalently, arbitrarily small (continuous) violations of the Pauli principle imply new degrees of freedom (new coherent quantal states of the particle). By transferring the notion of identity from 'particle' to quantal state, quantum mechanics achieves, via coherent mixing, states that continuously combine, simultaneously, elements of both identity and non-identity (the two-slit problem in a different guise).

To demonstrate the consistency of our viewpoint, we will construct a model (with a new degree of freedom) and show that in this way we can reproduce exactly the algebra of [1]. In particular this will show that the consequences found in [1] can be physically significant but need not be interpreted as violations of the Pauli principle.

We remark that the algebraic properties of our model (the Jordan pair construction) are of independent interest.

Let us denote by b and $b^\dagger$ the annihilation and creation operators for a field B which is such that the one-particle state is a mixture of a one-electron state $|\mathrm{e}\rangle$ and a one-muon state $|\mu\rangle$. (The notation e and μ is suggestive of lepton family number violation (which we discuss below), but for the moment e and μ are merely arbitrary labels.) As in [1], we disregard momentum and spin variables. Accordingly we have

$$b^\dagger|0\rangle = \cos\theta_B|\mathrm{e}\rangle + \sin\theta_B|\mu\rangle. \tag{1}$$

This field creates also a two-particle state $|2\rangle$, which—in this simple model with no momentum and spin variables—can only be a tensor product of an $|\mathrm{e}\rangle$ state and a $|\mu\rangle$ state—two $|\mathrm{e}\rangle$ and two $|\mu\rangle$ being forbidden by the Pauli principle:

$$\begin{aligned} b^\dagger|1\rangle &= 2\cos\theta_B \sin\theta_B|\mathrm{e}\rangle|\mu\rangle \\ &= \sin 2\theta_B|2\rangle. \end{aligned} \tag{2}$$

Here $\sin 2\theta_B$ is the parameter which corresponds to the Pauli-violating parameter β introduced by Greenberg and Mohapatra [2] and by Ignatiev and Kuzmin [1]. We thus require θ_B to be very small. A three-particle state is not allowed because of the Pauli principle. Therefore we obtain

$$b^\dagger|0\rangle = |1\rangle \qquad b^\dagger|1\rangle = \beta|2\rangle \qquad b^\dagger|2\rangle = 0 \qquad \beta = \sin 2\theta_B \tag{3}$$

and we also have

$$b|0\rangle = 0 \qquad b|1\rangle = |0\rangle \qquad b|2\rangle = \beta|1\rangle. \tag{4}$$

The third equation follows from the fact that b is the HC of $b^\dagger$. In fact suppose $b|2\rangle = \alpha|1\rangle$, then

$$\begin{aligned} \langle 2b^\dagger 1\rangle &= (b|2\rangle)^\dagger|1\rangle = \bar{\alpha} \\ \langle 2b^\dagger 1\rangle &= \beta\langle 2|2\rangle = \beta. \end{aligned} \tag{5}$$

Hence $\bar{\alpha} = \beta$; but β is real so $\alpha = \beta$. The operators b and $b^\dagger$ are not required to obey any anticommutation relation, as pointed out by Ignatiev and Kuzmin [1]. However, the following cubic relations hold for such operators:

$$\begin{aligned} &b^2 b^\dagger + \beta^2 b^\dagger b^2 = \beta^2 b \\ &b^2 b^\dagger + b^\dagger b^2 + b b^\dagger b = \mathrm{Tr}(b b^\dagger) b \end{aligned} \tag{6}$$

with $\mathrm{Tr}(bb^\dagger) = 1 + \beta^2$, plus the HC relations. We also have $b^3 = b^{\dagger 3} = 0$. These relations are equivalent to the ones found by Ignatiev and Kuzmin. The algebraic relations (6), satisfied by the operators b and $b^\dagger$ are not bilinear but *quadratic in b and linear in $b^\dagger$*. This is a clear indication that we are dealing with an underlying *Jordan pair structure.*

The notion of a Jordan pair provides a natural framework for the algebra described by (6) together with $b^3 = b^{\dagger 3} = 0$. This suggests a new way to look at this subject and to find generalisations.

A Jordan pair is a pair of moduli [8, 9, 10] $V = (V^+, V^-)$ which act on each other through the quadratic maps $U_{x^+}: V^- \to V^+$, $U_{x^-}: V^+ \to V^-$ such that

(i) $U_{x^+} y^-$ is quadratic in x^+ and linear in y^-,
(ii) $V_{x^+, y^-} U_{x^+} = U_{x^+} V_{y^-, x^+}$,
(iii) $V_{U_{x^+} y^-, y^-} = V_{x^+, U_{y^-} x^+}$,
(iv) $U_{U_{x^+} y^-} = U_{x^+} U_{y^-} U_{x^+}$,

where

$$V_{x^+, y^-}(z^+) = (U_{x^+ + z^+} - U_{x^+} - U_{x^+})(y^-) \tag{7}$$

is the linearisation of U_{x^+}. The same relations hold with the signs interchanged.

A simple example of a Jordan pair is given by the rectangular matrices. Suppose V^+ is the set of $m \times n$ matrices and V^- the set of $n \times m$ matrices; how can we define a multiplication which maps (V^+, V^-) onto itself? The answer is: use the quadratic map $U_{x^+} y^- = x^+ y^- x^+$, $x^+ \in V^+$, $y^- \in V^-$. It is straightforward to check that the properties (i) through (iv) defining Jordan pairs are satisfied by such a map. The notion of Jordan pairs also extends the concept of Jordan algebras.

The Jordan pair (V^+, V^-), which corresponds to (6) consists of 3×3 matrices, and their Hermitian conjugates, and of a quadratic map

$$U_{x^+} y^- = \mathrm{Tr}(x^+ y^-) x^+. \tag{8}$$

The special matrices b and $b^\dagger$ belong to V^+ and V^- respectively and are defined by the relations (6). The second relation involves directly $U_b b^\dagger$.

In the Jordan pair language the interaction Hamiltonian of [1] is

$$U_b b^\dagger + U_{b^\dagger} b. \tag{9}$$

Hence it can be written as

$$\mathrm{Tr}(bb^\dagger)(b + b^\dagger). \tag{10}$$

One can represent the Hamiltonian as a 3×3 Hermitian matrix which consists of the off-diagonal part (10) and the diagonal matrix N, which represents the number operator [1]. One can also relate N, quite naturally, to the Jordan pair structure by noticing that the commutation of N and b and $b^\dagger$ is an inner derivation of the Jordan pair.

The inner derivation [6] $\delta(x, y)$ defined by an element (x, y) of a Jordan pair V is

$$\delta(x, y) \doteq (V_{x,y}, -V_{y,x}) \tag{11}$$

and it is such that, in analogy with the standard definition for an algebra,

$$\mathrm{Id} - \varepsilon\delta(x, y) \tag{12}$$

is an automorphism with infinitesimal parameter ε.

It is important to stress that the minus sign in (11) is essential in order to yield the automorphism (12), and holds in general for any Jordan pair. It is precisely this minus sign which allows us to reproduce the standard commutation relations [1] of N with b and $b^\dagger$.

The appearance of a trace in the definition of $U_b b^\dagger$ suggests that the second of the relations (6) is related to the anticommutation relation of ordinary field theory. In quantum field theory for half-integer spin the creation and annihilation operators are defined on the Fock space F which is the direct sum of antisymmetric tensor product of Hilbert spaces. The definition is [11 and references therein], omitting the spin variables,

$$[b(f)\phi]_n(p_1, \ldots, p_n) = \int f(p)\phi_{n+1}(p, p_1, \ldots, p_n)(\mathrm{d}p) \tag{13}$$

$$[b^\dagger(f)\phi]_n(p_1, \ldots, p_n) = \sum_{j=1}^{n} (-1)^{j+1}\phi_{n-1}(p_1, \ldots, \hat{p}_j, \ldots p_n) \times f(p_j) \tag{14}$$

where $\phi \in F$, f is a test function, n refers to the n-particle component of a vector in F.

It follows that

$$[b(f), b^\dagger(\bar{g})]_+ = (g, f). \tag{15}$$

In the present case we can define the field B, which is the mixing of the eletron field and the muon field, as acting on the Fock space:

$$F = F_e \times F_\mu$$

where F_e and F_μ are the usual Fock spaces for the electron and the muon, and F is their tensor product (which we need not symmetrise or antisymmetrise). The definition of the annihilation and creation operators is not straightforward. We know that a consistent formulation cannot lead to the standard anticommutation relations; therefore the definitions of b and $b^\dagger$ cannot be the standard definitions (13) and (14). In fact if we applied such definitions straightforwardly we would not get

$$b|2\rangle = \beta|1\rangle$$

which we need to have for a constituent definition of the field B. The relation that takes the place of the relation (15) is suggested by the definition of the quadratic map of the Jordan pair:

$$U_b b^\dagger = \mathrm{Tr}(bb^\dagger)b. \tag{16}$$

Instead of the anticommutator we have a quadratic map typical of a general Jordan pair. As the analogue of the scalar product, we have the trace. For Jordan algebras, which are in a sense a special case of Jordan pairs, one has two equivalent notions, quadratic map and symmetric bilinear product. Thus, these Jordan algebras form a bridge to the more general concept of Jordan pairs, where, as in our case, one has only a quadratic map.

Currently we are investigating this type of field further.

Let us summarise. We have developed above an algebraic model reproducing the algebra, and physical consequences of [1], without, however, any violation of the Pauli principle. The model ascribes to the 'electron' a new degree of freedom (two states e and μ). If we take the model literally, as the actual electron and muon, then the relevant quantum number is the family label and the mixing we require is simply a violation of lepton family number (see e.g. [12] and references therein). Clearly a tiny interaction mixing muons into electron states would lead to a *triplet* spin component in the helium atom ground state, signalling physically an apparent (but not actual) violation of the Pauli principle. The essence of our model is that lepton family number violation makes significant for charged leptons an analogue to the Kobayashi–Maskawa mixing matrix. This implies interesting new physics, but our purpose here is only to validate our assertion: the Pauli principle is unique and cannot be truly violated without engendering unacceptable paradoxes. Any *apparent* violation is physically significant and implies new degrees of freedom.

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